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# THE SYNTHESIS OF BILINEAR SYSTEMS WITH DELAYED CONTROL†

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The optimal control of bilinear systems with delayed control and coordinates is synthesized by reduction to a solution of a linear boundary-value problem. A biological example is given. The optimal control of bilinear systems where only the phase coordinates were delayed was studied in [1].

## 1. STATEMENT OF THE PROBLEM

CONSIDER a bilinear system with after-effect

$$\dot{X}(t) = A_0(t)X(t) + A_1(t)X(t-h) + A_2(t)u(t-h_1) +$$

$$+ A(t)X(t) + B(t)u(t), \quad 0 \leqslant t \leqslant T, \; X(t) \in R^n, \; u \in R \tag{1.1}$$

The elements $A$, $A_i$ and $B$ are piecewise-continuous bounded functions, the delays $h$ and $h_1$ are positive, and the time $T > 0$ is specified.

The initial conditions have the form

$$X_0 = \varphi \in C[-h, 0], \quad u_0 = \psi \in D[-h_1, 0] \tag{1.2}$$

where $C[-h, 0]$ is the space of continuous functions in the interval $[-h, 0]$, $D[-h_1, 0]$ is the space of piecewise-continuous bounded functions with uniform metric, the functions $\varphi$ and $\psi$ are specified; $X_t = X(t+\theta)$, $-h \leqslant \theta \leqslant 0$; $u_t = u(t+\zeta)$, $-h_1 \leqslant \zeta \leqslant 0$.

System (1.1) can be considered for $A_0(t) \equiv 0$ if one makes the change of variable $X \equiv Z(t, 0)Y$, where $Z(t, s)$ is the matrix of the Cauchy equation $X^{\bullet}(t) = A_0(t)X(t)$. Henceforth we assume that $A_0 = 0$.

The quality criterion has the form

$$J = X'(T)N_1X(T) + \int_{t_0}^{T} [X'(t)N_2(t)X(t) + u'(t)N_0(t)u(t) +$$

$$+ F(t, X_t, u_t)]\, dt, \quad N_0 > 0 \tag{1.3}$$

where the prime denotes transposition and the $N_i$ are non-negative definite piecewise-continuous bounded matrices.

The continuous non-negative functional

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$$F: R \times C[-h, 0] \times D[-h_1, 0] \to R$$

the specific form of which is derived below, is obtained using the generalized work criterion [2], modified for systems with delay [3]; it enables one to reduce the Bellman equation to a linear equation.

An admissible control is any piecewise-continuous function $\mathbf{u}:[t_0, T] \to R$ such that for any $(t, \varphi, \psi) \in [t_0, T] \times C[-h, 0] \times D[-h_1, 0]$ a solution of Eq. (1.1) exists in the interval $[t, T]$ with initial conditions $X_t = \varphi$, $u_t = \psi$ under the control $\mathbf{u}$. The set of all admissible controls is denoted by $W$.

The optimal control problem consists of determining the admissible control which minimizes the quality criterion (1.3).

## 2. OPTIMALITY CONDITIONS

We determine the Bellman functional $V:[t_0, T] \times C[-h, 0] \times D[-h_1, 0] \to R$ in the following manner. Suppose $X(\cdot\,; t, \varphi, \psi; u):[t, T] \to R^n$, $(t \in [t_0, T],\ \varphi \in C[-h, 0],\ \psi \in D[-h_1, 0],\ u \in W)$ is a solution of Eq. (1.1) under the control $u$ and with initial conditions $X(t+\vartheta;\ t, \varphi, \psi;\ u) = \varphi(\theta)$, $-h \leqslant \theta \leqslant 0$, $u(t+\zeta) = \psi(\zeta)$, $-h_1 \leqslant \zeta \leqslant 0$.

We put $X_\tau(t, \varphi, \psi;\ u) = X(\tau + \cdot\,;\ t, \varphi, \psi;\ u):[-h, 0] \to R^n$.

Then

$$V(t, \varphi, \psi) = \inf_{U \in W} \{X'(T; t, \varphi, \psi; u)\ N_1 X(T; t, \varphi, \psi; u) +$$

$$+ \int_t^T X'(s; t, \varphi, \psi; u) N_2(s) X(s; t, \varphi, \psi; u) + u'(s) N_0(s) u(s) +$$

$$+ F(s, X_s(t, \varphi, \psi; u), u_s) ds\}$$

We note that the Bellman functional does not usually depend on the control. However, for a system of the form (1.1) with delay in the control device the minimum value of the quality criterion (1.3) depends uniquely on the preceding values of the control.

We introduce the operator

$$L_u V(t, \varphi, \psi) =$$

$$= \overline{\lim_{\Delta \to 0+}} \frac{1}{\Delta} [V(t+\Delta, X_{t+\Delta}(t, \varphi, \psi; u), v_{t+\Delta}) - V(t, \varphi, \psi)]$$

Here $v_{t+\Delta}$ is a control equal to $u$ in the interval $[t, t+\Delta]$ and equal to $\psi$ in the interval $[t+\Delta-h, t]$. We note that the operator $L_u V$ is the total derivative of the functional $V(t, \varphi, \psi)$ along the trajectory of system (1.1) under the control $u$.

Standard application of the dynamical programming method leads to the following optimality conditions.

*Theorem*. Suppose there exists a functional

$$V_0 : [t_0, T] \times C[-h, 0] \times D[-h_1, 0] \to R$$

satisfying a local Lipschitz condition, and a functional

$$u_0 : [t_0, T] \times C[-h, 0] \times D[-h_1, 0] \to R$$

satisfying the Carathéodory condition, such that

$$\inf_{u \in R} \Phi(u; t, \varphi, \psi) = \Phi(u_0; t, \varphi, \psi) \tag{2.1}$$

$$\Phi(u; t, \varphi, \psi) = [L_u V_0(t, \varphi, \psi) + \varphi'(0)N_2(t)\varphi(0) + u'N_0(t)u + F(t, \varphi, \psi)]$$
$$V_0(t, \varphi, \psi) = \varphi'(0)N_1(t)\varphi(0) \tag{2.2}$$

Then $u_0(t, \varphi, \psi)$ is the optimal control, and $V_0(t, \varphi, \psi)$ is the Bellman functional in problem (1.1)–(1.3).

*Remark*. The given optimality conditions also remain true when $u \in R^m$. Here $A$ is a tensor with components $a^i_{jl}$, the product $AXu$ is a vector with components

$$\sum_{j,l} a^i_{jl} \quad (i = 1, \ldots, n),$$

and the infimum in (2.1) is calculated over the vector parameter $u \in R^m$.

## 3. CONSTRUCTION OF THE SOLUTION

We will look for a solution of problem (2.1), (2.2) in the form

$$V_0(t, \varphi, \psi) = \varphi'(0)P(t)\varphi(0) + \varphi'(0)\int_{-h}^{0} Q(t, \tau)\varphi(\tau)d\tau +$$
$$+ \int_{-h}^{0} \varphi'(\tau)Q'(t, \tau)d\tau\varphi(0) + \int_{-h}^{0}\int_{-h}^{0} \varphi'(\tau)R(t, \tau, \tau_1)\varphi(\tau_1)d\tau d\tau_1 +$$
$$+ \varphi'(0)\int_{-h_1}^{0} L(t, \rho)\psi(\rho)d\rho + \int_{-h_1}^{0} \psi'(\rho)L'(t, \rho)d\rho\varphi(0) +$$
$$+ \int_{-h}^{0}\int_{h_1}^{0} \varphi'(\tau)K(t, \tau, \rho)\psi(\rho)d\tau d\rho + \int_{-h}^{0}\int_{-h_1}^{0} \psi'(\rho)K'(t, \tau, \rho)\varphi(\tau)d\tau d\rho +$$
$$+ \int_{-h_1}^{0}\int_{-h_1}^{0} \psi'(\rho)M(t, \rho, \rho_1)\psi(\rho_1)d\rho d\rho_1, \quad P(t) \geqslant 0 \tag{3.1}$$

All matrices in (3.1) are assumed to be piecewise-continuously differentiable bounded functions. We substitute (3.1) into expression (2.1) and find the control $u_0$ giving the infinum. We obtain

$$u_0(t, \varphi, \psi) = -N_0^{-1}(t)\,\{[C'(t)P(t) + L'(t, 0)]\,\varphi(0) +$$
$$+ \int_{-h}^{0} [K'(t, \tau, 0) + C'(t)Q(t, \tau)]\,\varphi(\tau)d\tau + \int_{-h_1}^{0} [M'(t, \tau, 0) + C'(t)L(t, \tau)]\,\psi(\rho)d\rho\} \tag{3.2}$$
$$C(t) = (A(t)\varphi(0) + B(t))$$

In functional (2.1) we now put

$$F(t, X_t, u_t) = N_0^{-1}(t)\{[C'(t)P(t) + L'(t, 0)]\,X(t) + \int_{-h}^{0} [K'(t, \tau_1, 0) +$$
$$+ C'(t)Q(t, \tau)]\,X(t+\tau)d\tau + \int_{-h_1}^{0} [M'(t, \tau, 0) + C'(t)L(t, \tau)]\,u(t+\tau)d\tau\}^2 \tag{3.3}$$

To determine the coefficients of the functional $V_0$ we substitute (3.1)–(3.3) into (2.1) and (2.2) and set to zero the coefficients in front of the corresponding quadratic forms of the variables $\varphi$ and $\psi$. We obtain the system of equations

$$P'(t)+Q(t,0)+Q'(t,0)+N_2(t)=0$$
$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \tau}\right)Q(t,\tau)+R(t,0,\tau)=0 \tag{3.4}$$
$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \tau}-\frac{\partial}{\partial \tau_1}\right)R(t,\tau,\tau_1)=0,\ \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \tau}-\frac{\partial}{\partial \rho}\right)K(t,\tau,\rho)=0$$
$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \rho}-\frac{\partial}{\partial \rho_1}\right)M(t,\rho,\rho_1)=0,\ \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \rho}\right)L(t,\rho)+K(t,0,\rho)=0$$
$$0\leqslant t\leqslant T,\ -h_1\leqslant\rho,\ \rho_1\leqslant 0,\ -h\leqslant\tau,\tau_1\leqslant 0$$

Setting to zero the quadratic forms of the variables $X(t-h)$ and $u(t-h_1)$ we obtain the boundary conditions

$$P(T)=N_1;\ Q(T,\ \tau)=R(T,\tau,\tau_1)=L(T,\rho)=K(T,\tau,\rho)=M(T,\rho,\rho_1)\equiv 0$$
$$-h_1<\rho,\rho_1\leqslant 0,\ -h<\tau,\tau_1\leqslant 0;$$
$$-Q(t,-h)+P(t)A_1(t)=0,\quad R(t,\tau,\rho)=R'(t,\rho,\tau)$$
$$-R(t,-h,\tau)-R'(t,\tau,-h)+2A_1'(t)Q(t,\tau)=0 \tag{3.5}$$
$$-L(t,-h_1)+P(t)A_2(t)=0,\ \ -K(t,-h,\rho_1)+A_1'(t)L(t,\rho_1)=0$$
$$-K(t,\tau,-h_1)+Q'(t,\tau)A_2(t)=0$$
$$-M(t,-h_1,\rho)-M'(t,\rho,-h_1)+2A_2'(t)L(t,\rho)=0$$
$$M(t,\rho,\rho_1)=M'(t,\rho_1,\rho),\ 0\leqslant t\leqslant T,\ -h_1\leqslant\rho,\rho_1\leqslant 0,\ -h\leqslant\tau,\tau_1\leqslant 0$$

Under the assumptions adopted a unique solution of problem (3.4), (3.5) exists [4] in the class of piecewise-continuously differentiable bounded functions and $P(t)\geqslant 0$. Hence [3] in some neighbourhood of the initial point $t=0$ a solution of problem (1.1)–(1.3) under the control (3.2) exists. This solution can be extended over the entire interval $[0,\ T]$ (which indicates the admissibility of the control $u_0$, and consequently, its optimality).

Indeed, since by (2.1) the total derivative of the functional $V_0$ is non-positive under control $u_0$ because of system (1.1), we have

$$V_0(t,X_t,u_{0t})\leqslant V_0(0,\varphi,\psi)$$

From this and from (2.1) it follows that

$$\int_0^t u_0'(s)N_0(s)u_0(s)ds\leqslant 2V_0(0,\varphi,\psi),\ u_0(s)=u_0(s,X_s,u_{0s})$$

Since the matrix $N_0(s)$ is uniformly positive definite, then for some constant $C$

$$\int_0^t u_0^2(s)ds\leqslant CV_0(0,\varphi,\psi)$$

Thus, in any interval $[0,\ t]$ in which a solution of problem (1.1)–(1.3) under control $u_0$ exists, this control is uniformly square-integral with respect to $t$ along the solution. Then the trajectory of $X$ corresponding to $u_0$ is a solution of the linear equation (1.1) (with $u$ replaced by $u_0$) whose coefficients are square-integrable in $[0, T]$. Hence the solution $X(t)$ of problem (1.1)–(1.3) under the control $u_0$ can be extended over the entire interval $[0, T]$.

Thus, the solution of the original optimal control problem reduces to the boundary-value problem (3.4), (3.5) and is given by formulae (3.1) and (3.2).

We will first of all consider a special case in which the solution of problem (3.4), (3.5) can be represented in analytic form. We assume that system (1.1) only has a delay $h_1$ in the control

$$X^{\cdot}(t) = [A(t)X(t) + B(t)]\,u(t) + A_2(t)u(t-h_1)$$

In this case, the solution of the corresponding Bellman equation (2.1), (2.2) has the form

$$V_0(t, \varphi, \psi) = \varphi'(0)P(t)\varphi(0) + \varphi'(0)\int_{-h_1}^{0} L(t, \rho)\psi(\rho)d\rho +$$
$$+ \int_{-h_1}^{0} \psi'(\rho)L'(t, \rho)d\rho\varphi(0) + \int_{-h_1}^{0}\int_{-h_1}^{0} \psi'(\rho)M(t, \rho, \rho_1)\psi(\rho_1)d\rho d\rho_1$$

The optimal control $u_0(t, X_t, U_{0t})$ by virtue of (3.2) is given by the formulae

$$U_0(t, X_t, U_{0t}) = -N_0^{-1}(t)[(A(t)X + B(t)]'(P(t)X(t) +$$
$$+ \int_{-h_1}^{0} L(t, \rho)U_0(t+\rho)d\rho) + \int_{-h_1}^{0} M(t, 0, \rho)U_0(t+\rho)d\rho + L'(t, 0)X(t)]$$

The matrices $P$, $L$ and $M$ are defined as solutions of the equations

$$\begin{aligned} &P^{\cdot}(t) + N(t) = 0, \quad 0 \leqslant t \leqslant T \\ &\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \rho}\right)L(t, \rho) = 0, \quad -h_1 \leqslant \rho, \rho_1 \leqslant 0 \\ &\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \rho} - \frac{\partial}{\partial \rho_1}\right)M(t, \rho, \rho_1) = 0 \end{aligned} \tag{3.6}$$

with boundary conditions

$$\begin{aligned} &P(T) = N_1, \; L(T, \rho) = 0, \; M(t, \rho, \rho_1) = 0, \; 0 \leqslant t \leqslant T, \; -h_1 < \rho, \rho_1 \leqslant 0 \\ &P(t)A_2(t) - L(t, -h_1) = 0 \\ &2A_2'(t)L(t, \rho) - M(t, -h_1, \rho) - M'(t, \rho, -h_1) = 0 \end{aligned} \tag{3.7}$$

The solution of the boundary-value problem (3.6), (3.7) has the form

$$P(t) = N_1 + \int_t^T N_2(s)ds, \quad 0 \leqslant t \leqslant T$$

$$L(t, \rho) = \begin{cases} P(t+\rho+h_1)A_2(t+\rho+h_1), & t+\rho+h_1 < T, \\ 0 & , t+\rho+h_1 \geqslant T, \end{cases}$$

$$M(t, \rho, \rho_1) = \begin{cases} A_2'(t+\rho+h_1)P(t+\beta+h_1)A_2(t+\rho+h_1), & \omega < 0 \\ 0 & , \quad \omega \geqslant 0 \end{cases}$$

$$\omega = \max(t+\rho+h_1-T,\ t+\rho_1+h_1-T),\ \beta=\max(\rho,\rho_1)$$

We now turn to the general case (3.5), (3.5). We represent the solution of problem (3.4), (3.5) as the sum of two solutions: the first is for $N_2=0$ and an arbitrary matrix $N_1 \geqslant 0$, and the second is for $N_1=0$ and an arbitrary matrix $N_2 \geqslant 0$.

The first solution (for $N_2=0$) has the form

$$\begin{aligned}
&P(t)=b'(t)N_1 b(t), \quad Q(t,\tau)=-b'(t)N_1 b^{\cdot}(t+\tau)\\
&R(t,\tau,\tau_1)=b'^{\cdot}(t+\tau)N_1 b^{\cdot}(t+\tau_1),\quad 0\leqslant t\leqslant T,\quad -h\leqslant\tau,\tau_1\leqslant 0\\
&L(t,\rho)=b'(t)N_1 b_1(t+\rho+h_1), K(t,\tau,\rho)=-b'(t+\tau)N_1 b_1(t+\rho+h_1)\\
&M(t,\rho,\rho_1)=b_1'(t+\rho+h_1)N_1 b_1(t+\rho_1+h_1)\\
&b_1(t+\rho+h_1)=\begin{cases} b(t+\rho+h_1)A_2(t+\rho+h_1), & t+\rho+h_1\leqslant\min(T,\rho+h_1)\\ 0, & t+\rho+h_1>\min(T,\rho+h_1)\end{cases}
\end{aligned}\tag{3.8}$$

The matrix $b(t)$ is a solution of the Cauchy problem

$$b^{\cdot}(t)=-b(t+h)A_1(t+h),\qquad b(T)=I,\qquad b(s)\equiv 0,\ \ s\geqslant T$$

To construct the second solution (for $N_1=0$) we put $t_i=T-ih$ $(i=0,1,\ldots)$ and determine the functions $L_1$, $M_1$, $A_3$ and $A_4$. For $t_{i+1}\leqslant t\leqslant t_i$ the function $A_3(t+\tau+h)=0$ if $-t+t_{i+1}<\tau\leqslant 0$ and $A_3(t+\tau+h)=A_1(t+\tau+h)$ if $-h\leqslant\tau\leqslant -t+t_{i+1}$. Similarly, $A_4(t+\rho+h_1)=A_2(t+\rho+h_1$, $-h_1\leqslant \rho\leqslant -t+t_{i+1}$; $A_4(t+\rho+h_1)=0$ if $-t+t_{i+1}<\rho\leqslant 0$, $t_{i+1}\leqslant t\leqslant t_i$. Finally, $M_1(t_i,\ t+\rho-t_i,\ \rho)=0$, $L_1(t_i,\ t+\rho-t_i)=0$, $K_1(t_i,\ \tau,\ t+\rho-t_i)=0$ if $-h_1\leqslant\rho<-t+t_{i+1}$, and $K_1(t_i,\ \tau,\ t+\rho-t_i)=K(t_i,\ \tau,\ t+\rho-t_i)$, $M_1(t_i,\ t+\rho-t_i,\ \rho)=M(t_i,\ t+\rho-t_i,\ \rho)$, $L_1(t_i,\ t+\rho-t_i)=L(t_i,\ t+\rho-t_i)=0$ if $-t+t_{i+1}\leqslant\rho\leqslant 0$, $t_{i+1}\leqslant t\leqslant t_i$.

Then for $t_{i+1}\leqslant t\leqslant t_i$ we have the recursive relations

$$\begin{aligned}
&P(t)=P(t_i)+\int_t^{t_i}N_2(s)ds+\int_{t-t_i}^{0}[Q(t_i,s)+Q'(t_i,s)]\,ds+\int_{t-t_i}^{0}\int_{t-t_i}^{0}R(t_i,s,\alpha)dsd\alpha\\
&Q(t,\tau)=[P(t_i)+\int_{t+\tau+h}^{t_i}N_2(s)ds+\int_{t-t_i}^{0}Q(t_i,s)ds+\int_{t-t_{i+\tau}}^{0}Q'(t_i,s)ds+\\
&+\int_{t-t_i}^{0}d\alpha\int_{t-t_{i+1}+\tau}^{0}R(t_i,s,\alpha)ds]A_3(t+\tau+h)+Q_1(t_i,t+\tau-t_i)+\int_{t-t_i}^{0}R_1(t_i,s,t+\tau-t_i)ds
\end{aligned}\tag{3.9}$$

Here

$$\begin{aligned}
&Q_1(t_i,\ \tau+t-t_i)=Q(t_i,\tau+t-t_i)\\
&R_1(t_i,\tau+t-t_i,\tau_1)=R(t_i,\tau+t-t_i,\tau_1), -t+t_{i+1}\leqslant\tau\leqslant 0 \text{ and } Q_1(t_i,\tau+t-t_i)=0\\
&R_1(t_i,\tau+t-t_i,\tau_1)=0,\ -h\leqslant\tau\leqslant -t+t_{i+1};\ -h\leqslant\tau_1\leqslant 0,\ t_{i+1}\leqslant t\leqslant t_i\\
&R_2(t_i,\tau+t-t_i,\tau_1+t-t_i)=\begin{cases} R(t_i,\tau+t-t_i,\rho+t-t_i),\ t_{i+1}\leqslant t\leqslant t_i, -t+t_{i+1}\leqslant\tau,\tau_1\leqslant 0\\ 0, \text{ if at least one of the arguments does not lie in the interval } [-t+t_{j+1},0]\end{cases}
\end{aligned}$$

Then

$$R(t,\tau,\tau_1)=A_3'(t+\tau+h)[\int_{t+h+\max(\tau,\tau_1)}^{t_i}N_2(s)ds+P(t_i)+\int_{t-t_{i+1}+\tau_1}^{0}Q'(t_i,s)ds+$$

$$+\int_{t-t_{i+1}+\tau}^{0} Q(t_i,s)ds+\int_{t-t_{i+1}+\tau}^{0} d\alpha \int_{t-t_{i+1}+\tau_1}^{0} R(t_i,s,\alpha)ds]A_3(t+\tau_1+h)+$$

$$+A_3'(t+\tau+h)[Q_1'(t_i,t+\tau_1-t_i)+\int_{t-t_{i+1}+\tau}^{0} R_1(t_i,s,t+\tau_1-t_i)ds]+$$

$$+[Q_1(t_i,t+\tau-t_i)+\int_{t-t_{i+1}+\tau_1}^{0} R_1(t_i,t+\tau-t_i,s)ds]A_3(t+\tau_1+h)+$$

$$+R_2(t_i,t+\tau-t_i,t+\tau_1-t_i) \tag{3.10}$$

We then have

$$L(t,\rho)=[\int_{t+\rho+h_1}^{t_i} N_2(s)ds+P(t_i)+\int_{t-t_i}^{0} Q'(t_i,s)ds+\int_{t-t_{i+1}+\rho}^{0} Q(t_i,s)ds+$$

$$+\int_{t-t_i}^{0} d\alpha\int_{t-t_{i+1}+\rho}^{0} R(t_i,s,\alpha)ds]A_4(t+\rho+h_1)+L_1(t_i,t+\rho-t_i)+$$

$$+\int_{t-t_i}^{0} K_1(t_i,s,t+\rho-t_i)ds \tag{3.11}$$

$$K(t,\tau,\rho)=A_3'(t+\tau+h)[\int_{t+\max(\tau+h,\rho+h_1)}^{t_i} N_2(s)ds+P(t_i)+\int_{t-t_{i+1}+\rho}^{0} Q(t_i,s)\,ds+$$

$$+\int_{t-t_{i+1}+\tau}^{0} Q'(t_i,s)ds+\int_{t-t_{i+1}+\tau}^{0} d\alpha\int_{t-t_{i+1}+\rho}^{0} R(t_i,s,\alpha)ds]A_4(t+\rho+h_1)+$$

$$+A_3'(t+\tau+h)[L_1(t_i,t+\rho-t_i)+\int_{t-t_{i+1}+\tau}^{0} K_1(t_i,s,t+\rho-t_i)ds]+$$

$$+[Q_1'(t_i,t+\tau-t_i)+\int_{t-t_{i+1}+\rho}^{0} R_1(t_i,s,t+\tau-t_i)ds]\times$$

$$\times A_4(t+\rho+h_1)+K_2(t_i,t+\tau-t_i,t+\rho+t_i) \tag{3.12}$$

Finally

$$M(t,\rho,\rho_1)=A_4'(t+\rho+h_1)[\int_{t+h_1+\max(\rho,\rho_1)}^{t_i} N_2(s)ds+P(t_i)+\int_{t-t_{i+1}+\rho}^{0} Q'(t_i,s)ds+$$

$$+\int_{t-t_{i+1}+\rho}^{0} Q(t_i,s)ds+\int_{t-t_{i+1}+\rho}^{0} d\alpha\int_{t-t_{i+1}+\rho_1}^{0} R(t_i,s,\alpha)ds]A_4(t+\rho_1+h_1)+A_4(t+\rho+h_1)\times$$

$$\times[L_1(t_i,t+\rho_1-t_i)+\int_{t-t_{i+1}+\rho}^{0} K_1(t_i,s,t+\rho_1-t_i)ds]+[L_1'(t_i,t+\rho-t_i)+$$

$$+\int_{t-t_{i+1}+\rho_1}^{0} K_1'(t_i,s,t+\rho-t_i)ds]A_4'(t+\rho_1+h_1)+M_2(t_i,t+\rho-t_i,t+\rho_1-t_i) \tag{3.13}$$

$$M_2(t_i,t+\rho-t_i,t+\rho_1-t_i)=\begin{cases} M(t_i,t+\rho-t_i,t+\rho_1-t_i),\ t_{i+1}\leqslant t\leqslant t_i,\ -t+t_{i+1}\leqslant\rho,\rho_1\leqslant 0 \\ 0, \text{ if at least one of the arguments } \rho,\ \rho_1 \\ \text{does not lie in the interval } [-t+t_{i+1},0] \end{cases}$$

The recurrence formulae (3.9)–(3.13) enable us to obtain a second solution of problem (3.4), (3.5) sequentially in the intervals $[t_{i+1}, t_i]$ $(i=0,1,\ldots)$. Adding this solution to (3.8), we obtain the general solution. In particular, for $T-h\leqslant t\leqslant T$ we have

$$P(t) = N_1 + \int_t^T N_2(r)dr, \quad Q(t, \tau) = P(t+\tau+h)A_3(t+\tau+h)$$

$$R(t, \tau, \tau_1) = A_3'(t+\tau+h)P(t+h+\max(\tau, \tau_1))A_3(t+\tau_1+h)$$

$$L(t, \rho) = P(t+\rho+h_1)A_4(t+\rho+h_1)$$

$$M(t, \rho, \rho_1) = A_4'(t+\rho+h_1)P(t+h_1+\max(\rho, \rho_1))A_4(t+\rho_1+h_1)$$

$$K(t, \tau, \rho) = A_3'(t+\tau+h)P(t+\max(h+\tau, h_1+\rho))A_4(t+\rho+h_1)$$

## 4. SOME GENERALIZATIONS

The results obtained can be modified to other controlled systems. We give one of these generalizations to a system of the form

$$X^{\cdot}(t) = A_1(t)X(t-h) + A_2(t)u(t-h_1) + (A(t, X_t) + B(t))u(t), \quad 0 \leq t \leq T \tag{4.1}$$

Here the functional $A: R \times C[-h_2, 0] \to R$ is measurable with respect to its set of arguments, piecewise-continuous with respect to $t$ and satisfies a Lipschitz condition for its second argument; $X_t$ in $A(t, X_t)$ denotes the section of the trajectory $X_t = X(t+\tau)$, $-h_2 \leq \tau \leq 0$.

The solution of Eq. (4.1) is governed by the initial conditions (1.3) and $X_0 = \varphi \in C[-\max(h_1, h_2), 0]$.

The quality criterion has the form (1.3) with functional $F$ given by (3.3), throughout which $C(t)$ is replaced by $A(t, X_t)$. The optimal control has the form (3.2) with $C(t)$ replaced by $A(T, \varphi)$. The Bellman functional (3.1) stays unchanged. In particular, it follows that although the optimal control and trajectory depend on $h_2$, the functional (3.2) does not depend on it. This means that for $h = 0$ the optimum value of the quality criterion does not in general depend on the values of the initial function $\varphi(s)$, $s < 0$, although both the trajectory and the optimal control depend on it strongly.

## 5. EXAMPLE

Consider a model of a controlled process of the microbiological growth of bacteria in a closed vessel under the condition that transfer of nutrients from one point to another occurs over a finite time together with the production of output material. Such a process is described by a bilinear model with delay in the control which is the rate of supply of nutrients into the bioreactor

$$\begin{aligned} m^{\cdot}(t) &= \gamma(t)m(t) - U(t)m(t) - m(t-\tau) + \mu_1(t)U(t-\rho) \\ S^{\cdot}(t) &= \frac{\gamma(t)}{K_1}m(t) - U(t)S(t) + S_r U(t) + \mu_2(t)U(t-\rho) \end{aligned} \tag{5.1}$$

Taking into account the interaction between the cellular material with the nutrient medium, the first equation characterizes the balance of microbiological mass in a closed vessel, and the second describes the production process for the product being synthesised. In the model (5.1) being considered we use the notation of [5]: $m(t)$ is the microbiological mass concentration of the bacterial culture, $S(t)$ is the concentration of output product, $U(t)$ is the rate of supply of nutrients required to support bacterial activity, $\gamma(t)$ is the growth rate coefficient of the cellular mass, $K_1$ is the growth rate parameter of the output product, $m(t-\tau)$ is a term describing the loss of bacterial activity after a finite time $\tau$, $S_r$ is some constant concentration of output product, $U(T-\rho)$ is the rate of supply of nutrients at an earlier time for supporting bacterial activity, and $\mu_1(t)U(t-\rho)$ and $\mu_2(t)U(t-\rho)$ are, respectively, the concentrations of nutrients in the biomass and in the output substratum at an earlier time.

At the initial time $t_0$ there is an injection of bacterial mass into the closed vessel, and so it is natural to

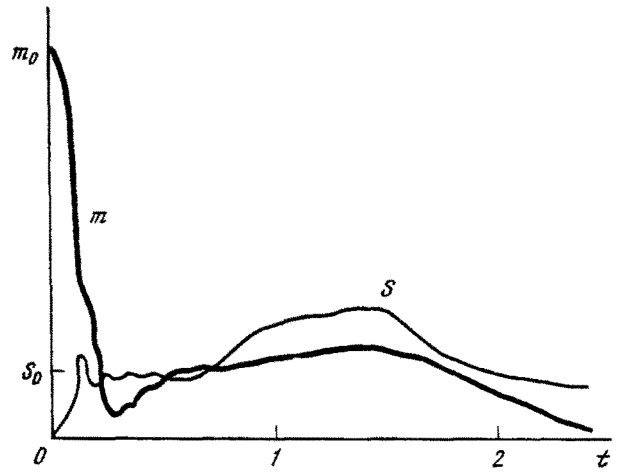


FIG. 1.

assume the absence of an output substratum and microbiological mass from the nutrient medium up to time $t_0$

$$
\begin{aligned}
&S(t_0) = 0\\
&m(t_0) = m_0, \quad m(t_0 + \theta) = 0, \; -\tau \le \theta < 0\\
&U(t_0 + \vartheta) = 0, \; -\rho \le \vartheta < 0
\end{aligned}
\tag{5.2}
$$

Under these conditions one should reach a specified level of output product $S_0$ in a finite time and with minimum consumption of nutrients. We choose a quality criterion appropriate to the problem in question, of the form

$$
J = \beta_1(S(T) - S_0)^2 + \beta_2 m(T)^2 + \int_{t_0}^{T} ((S(t) - S_0)^2 + \alpha U^2(t) + F(t, S, m_t, U_t))dt \tag{5.3}
$$

Figure 1 shows the phase trajectories for $m(t)$ and $S(t)$ under optimal control constructed by the method proposed for the modified quality functional.

The problem was solved numerically for the following parameter values

$$
\begin{aligned}
&t_0 = 0, \; T = 3, \; \tau = 1, \; S_r = 3.5, \; S_0 = 1.5, \; \rho = 0.2,\\
&K_1 = 52, \; \gamma(t) = 0{,}05, \; \beta_1 = \beta_2 = 1, \; \alpha = 1, \; m_0 = 10
\end{aligned}
$$

The graph shows that $S$ and $m$ tend to the specified values.

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